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The First Leap Zagreb Index of Some Graph Operations

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Abstract

Recently introduced leap Zagreb indices of a graph based on the second degrees of vertices (number of their second neighbors). The first leap Zagreb index $LM_1(G)$ is equal to the sum of squares of the second degrees of the vertices, the second leap Zagreb index $LM_2(G)$ is equal to the sum of the products of the second degrees of pairs of adjacent vertices of G and the third leap Zagreb index $LM_3(G)$ is equal to the sum of the products of the first degrees with the second degrees of the vertices. In this paper, exact expressions for the first Zagreb index of some graph operations containing the corona product, cartesian product, composition, disjunction and symmetric difference of graphs are presented.

Key words: Degree (of vertex), second degrees, Zagrab indices, first leap Zagrab index, graph operations.

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1 Introduction

In this paper, we are concerned only with simple graphs, i.e., finite graphs having no loops, multiple and directed edges. Let $G = (V, E)$ be such a graph with vertex set $V(G)$ and edges set $E(G)$. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. The distance $d_G(u, v)$ between any two vertices u and v of a graph G is equal to the length of (number of edges in) a shortest path connecting them. For a vertex $v \in V(G)$ and a positive integer k, the open k-neighborhood of v in a graph G is denoted by $N_k(v/G)$ and is defined as $N_k(v/G) = \{u \in V(G) : d_G(u, v) = k\}$. The k-distance degree of a vertex v in G is denoted by $d_k(v/G)$ (or simply $d_k(v)$ if no misunderstanding) and is defined as the

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number of k-neighbors of the vertex v in G, i.e., $d_k(v/G) = |N_k(v/G)|$. It is clearly that $d_1(v/G) = d(v/G)$ for every $v \in V(G)$. The complement $\overline{G}$ of a graph G is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

For a vertex v of G, the eccentricity $e(v) = \max\{d_G(v,u) : u \in V(G)\}$. The diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$ and the radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$. Let $H \subseteq V(G)$ be any subset of vertices of G. Then the induced subgraph $G\langle H\rangle$ of G is the graph whose vertex set is H and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in H. A graph G is called F-free graph if no induced subgraph of G is isomorphic to F. For any terminology or notation not mention here, we refer to [9].

A topological index of a graph is a graph invariant number calculated from a graph representing a molecule and applicable in chemistry. The zagreb indices have been introduced, more than forty four years ago, by Gutman and Trinajestic [7], in 1972, and elaborated in [6]. They are defined as:

$$M_1(G) = \sum_{v \in V(G)} d_1^2(v/G) \qquad and \qquad M_2(G) = \sum_{uv \in E(G)} d_1(u/G)d_1(v/G).$$

For properties of the two Zagreb indices see [2], [3], [6], [12], [16] and the papers cited therein.

In recent years, some novel variants of ordinary Zagreb indices have been introduced and studied, such as Zagreb coincides [1], [8], multiplicative Zagreb indices [5], [14], [16], multiplicative sum Zagreb index [4], [15], and multiplicative Zagreb coincides [17] and etc. The Zagreb coindices are defined as:

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} (d_1(u/G) + d_1(v/G)) \qquad and \qquad \overline{M_2}(G) = \sum_{uv \notin E(G)} d_1(u/G)d_1(v/G).$$

Recently, Naji et al. [10], have been introduced a new distance-degree-based topological indices conceived depending on the second degrees of vertices (number of their second neighbors), and are so-called leap Zagreb indices of a graph G and are defined as, respectively.

$$LM_1(G) = \sum_{v \in V(G)} d_2^2(v/G)$$

$$LM_2(G) = \sum_{uv\in E(G)} d_2(u/G)d_2(v/G)$$

$$LM_3(G) = \sum_{v\in V(G)} d(v/G)d_2(v/G).$$

From the above definitions, the following identities have been established.

$$LM_1(G) = \sum_{v\in V(G)} \sum_{u\in N_2(v/G)} d_2(u/G)$$

$$LM_2(G) = \frac{1}{2} \sum_{v\in V(G)} d_2(v/G) \sum_{u\in N(v/G)} d_2(u/G)$$

$$\begin{aligned} LM_3(G) &= \sum_{v\in V(G)} \sum_{u\in N(v/G)} d_2(u/G) \\ &= \sum_{v\in V(G)} \sum_{u\in N_2(v/G)} d_1(u/G) \\ &= \sum_{uv\in E(G)} \Big(d_2(u/G) + d_2(v/G)\Big). \end{aligned}$$

For properties of the leap Zagreb indices see [10].
In this paper, the explicit formulae for a first leap Zagreb index of some graph operations containing the corona product, cartesian product, composition, disjunction and symmetric difference of graphs will be presented.

The following fundamental results which will be required for many of our arguments in this paper are found in Yamaguchi [18] and Soner and Naji [13].

Theorem 1.1. ([13],[18]) Let G be a connected graph with n vertices and m edges. Then

$$d_2(v/G) \leq (\sum_{u\in N_1(v/G)} d_1(u/G)) - d_1(v/G)$$

and equality holds if and only if G is a $\{C_3, C_4\}$-free graph.

2 The first leap Zagreb index of graph operations

In this section, we present exact formulae for the first leap Zagreb index of corona product $G \circ H$, cartesian product $G \times H$, composition $G[H]$, disjunction $G \vee H$ and symmetric difference $G \oplus H$ of two graphs G and H. Throughout this section our notation is standard and taken mainly from [11].

2.1 Corona product

Definition 2.1. Let G and H be two graphs on disjoint vertex sets with n_1 and n_2 vertices, respectively. The corona $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n_1 copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

It is clear from the definition of $G \circ H$ that $n = |V(G \circ H)| = n_1 + n_1 n_2$ and $m = |E(G \circ H)| = m_1 + n_1(n_2 + m_2)$, where m_1 and m_2 are the sizes of G and H, respectively. In the following results, H^j, for $1 \leq j \leq n_1$, denotes the copy of a graph H which joining to a vertex v_j of a graph G. Note that in general this operation is not commutative.

Theorem 2.2. Let G_1 and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then
$LM_1(G_1 \circ H) = LM_1(G) + 2n_2 LM_3(G) + n_2(n_2+1)M_1(G) + n_1 M_1(H) + n_2(n_2-1)\Big[n_1 n_2(n_2 - 1) - 4n_1 m_2 + 4n_2 m_1\Big] - 8m_1 m_2.$

Proof: Since, for every $v \in V(G \circ H)$ either $v \in V(G)$ or $v \in V(H^j)$, for some $1 \leq j \leq n_1$, then to compute $d_2(v/G \circ H)$, we consider the following cases:

Case 1: If $v \in V(G)$, then $d_2(v/G \circ H) = d_2(v/G) + n_2 d(v/G)$.

Case 2: If $v \in V(H^j)$, then

$$d_2(v/G \circ H) = \sum_{j=1}^{n_1} d_2(v/\{v_j\} \circ H^j) = \sum_{j=1}^{n_1} \Big[(n_2 - 1) - d(v/H^j) + d(v_j/G)\Big].$$

Thus,

$$\begin{aligned}
LM_1(G \circ H) &= \sum_{v \in V(G \circ H)} d_2^2(v/G \circ H) \\
&= \sum_{v \in V(G)} d_2^2(v/G \circ H) + \sum_{j=1}^{n_1} \sum_{v \in V(H^j)} d_2^2(v/\{v_j\} \circ H^j) \\
&= \sum_{v \in V(G)} \Big[d_2(v/G) + n_2 d(v/G)\Big]^2 + \sum_{j=1}^{n_1} \sum_{v \in V(H^j)} \Big[(n_2 - 1) - d(v/H^j) + d(v_j/G)\Big]^2 \\
&= \sum_{v \in V(G)} \Big[d_2^2(v/G) + 2n_2 d(v/G) d_2(v/G) + n_2^2 d(v/G)\Big] \\
&\quad + \sum_{j=1}^{n_1} \sum_{v \in V(H^j)} \Big[(n_2 - 1)^2 - 2(n_2 - 1) d(v/H^j) + d^2(v/H^j)
\end{aligned}$$

$$\begin{aligned}
&+ 2(n_2-1)d(v_j/G) - 2d(v/H^j)d(v_j/G) + d^2(v_j/G)\Big]\\
&= \sum_{v\in V(G)} d_2^2(v/G) + 2n_2 \sum_{v\in V(G)} [d(v/G)d_2(v/G)] + n_2^2 \sum_{v\in V(G)} d(v/G)\\
&\quad + \sum_{j=1}^{n_1} \sum_{v\in V(H^j)} (n_2-1)^2 - 2(n_2-1)\sum_{j=1}^{n_1} \sum_{v\in V(H^j)} d(v/H^j)\\
&\quad + \sum_{j=1}^{n_1} \sum_{v\in V(H^j)} d^2(v/H^j) + 2(n_2-1)\sum_{j=1}^{n_1} \sum_{v\in V(H^j)} d(v_j/G)\\
&\quad - 2\sum_{j=1}^{n_1} \sum_{v\in V(H^j)} [d(v/H^j)d(v_j/G)] + \sum_{j=1}^{n_1} \sum_{v\in V(H^j)} d^2(v_j/G)\\
&= LM_1(G) + 2n_2LM_3(G) + n_2^2M_1(G) + n_1n_2(n_2-1)^2 - 4n_1m_2(n_2-1)\\
&\quad + n_1M_1(H) + 4n_2m_1(n_2-1) - 8m_1m_2 + n_2M_1(G)\\
&= LM_1(G) + 2n_2LM_3(G) + n_2(n_2+1)M_1(G) + n_1M_1(H)\\
&\quad + n_2(n_2-1)\Big[n_1n_2(n_2-1) - 4n_1m_2 + 4n_2m_1\Big] - 8m_1m_2.
\end{aligned}$$

■

2.2 Cartesian product

Definition 2.3. For given graphs G and H their Cartesian product, denoted $G\Box H$, is defined as the graph on the vertex set $V(G)\times V(H)$, and vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ of $V(G)\times V(H)$ are connected by an edge if and only if either ($u_1 = v_1$ *and* $u_2v_2 \in E(H)$) or ($u_2 = v_2$ *and* $u_1v_1 \in E(G)$).

It is a well known fact that the Cartesian product of graphs is commutative and associative up to isomorphism, $|V(G\Box H)| = |V(G)||V(H)|$, the distance between any two vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ in $G\Box H$ is given by $d_{G\Box H}(u,v) = d_G(u_1,v_1)+d_H(u_2,v_2)$.

The following result is required to prove the next main result.

Theorem 2.4. [13] Let G and H be connected graphs of orders n_1 and n_2, respectively. Then for any vertex $(u,v)\in V(G\Box H)$,

$$d_k((u,v)/G\Box H) = \sum_{i=0}^{k}\Big(d_i(u/G)\ d_{k-i}(v/H)\Big).$$

Theorem 2.5. Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then

$LM_1(G\Box H) = n_2LM_1(G)+4m_2LM_3(G)+M_1(G)M_1(H)+2xy+4m_1LM_3(H)+n_1LM_1(H)$

where $x = \sum_{u\in V(G)} d_2(u/G)$ and $y = \sum_{v\in V(H)} d_2(u/H)$.

Proof: Let G and H be connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then by Theorem 2.4,
$d_2((u,v)/G\Box H) = d_2(u/G) + d(u/G)d(v/H) + d_2(v/H)$, for every $u \in V(G)$ and $v \in V(H)$. Hence,

$$
\begin{aligned}
LM_1(G\Box H) &= \sum_{(u,v)\in V(G\Box H)} d_2^2((u,v)/G\Box H)\\
&= \sum_{u\in V(G)}\sum_{v\in V(H)} \Big(d_2(u/G) + d(u/G)d(v/H) + d_2(v/H)\Big)^2\\
&= \sum_{u\in V(G)}\sum_{v\in V(H)} \Bigg(d_2^2(u/G) + 2d(u/G)d_2(u/G)d(v/H) + d^2(u/G)d^2(v/H)\\
&\quad + 2d_2(u/G)d_2(v/H) + 2d(u/G)d(v/H)d_2(v/H) + d_2^2(v/H)\Bigg)\\
&= \sum_{u\in V(G)}\sum_{v\in V(H)} d_2^2(u/G) + 2\sum_{u\in V(G)}\sum_{v\in V(H)} \Big(d(u/G)d_2(u/G)d(v/H)\Big)\\
&\quad + \sum_{u\in V(G)}\sum_{v\in V(H)} \Big(d^2(u/G)d^2(v/H)\Big) + 2\sum_{u\in V(G)}\sum_{v\in V(H)} \Big(d_2(u/G)d_2(v/H)\Big)\\
&\quad + 2\sum_{u\in V(G)}\sum_{v\in V(H)} \Big(d(u/G)d(v/H)d_2(v/H)\Big) + \sum_{u\in V(G)}\sum_{v\in V(H)} d_2^2(v/H)\\
&= n_2LM_1(G) + 4m_2LM_3(G) + M_1(G)M_1(H)\\
&\quad + 2\Big(\sum_{u\in V(G)} d_2(u/G)\Big)\Big(\sum_{v\in V(H)} d_2(v/H)\Big) + 4m_1LM_3(H) + n_1LM_1(H).
\end{aligned}
$$

■

From Theorems 1.1 and 2.5, the following result follows:

Corollary 2.6. Let G and H be $\{C_3, C_4\}$-free connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then
$LM_1(G\Box H) = n_2LM_1(G)+4m_2LM_3(G)+n_1LM_1(H)+4m_1LM_3(H)+3M_1(G)M_1(H)-4m_1M_1(G) - 4m_2M_1(H) + 8m_1m_2$.

Corollary 2.7. For any two complete graphs K_n and K_p, $n, p \geq 2$,

$$LM_1(K_n \square K_p) = M_1(K_n)M_1(K_p).$$

Corollary 2.8. Let G be a $\{C_3, C_4\}$-free connected graph with $n \geq 3$ vertices and m edges, such that $G \not\cong K_n$. Then

$$LM_1(G \square K_p) = pLM_1(G) + 2p(p-1)LM_3(G) + p(p-1)^2 M_1(G).$$

As an application of the above result, we list explicit formulae for the first leap Zagreb index of $P_s \square P_t$, $P_s \square C_t$ and $C_s \square C_t$. These graphs are known as the rectangular grid, the C_4 nanotube, and the C_4 nanotorus, respectively. The formulae follow from Theorem 2.5 and Corollary 2.6, by bering in mind, for $n \geq 5$, $M_1(P_n) = 4n - 6$, $LM_1(P_n) = 4(n-3)$, $LM_3(P_n) = 4n - 10$ and $M_1(C_n) = LM_1(C_n) = LM_3(C_n) = 4n$.

Corollary 2.9. For $s, t \geq 5$,

(a) $LM_1(P_s \square P_t) = 60st - 100(s+t) + 132$

(b) $LM_1(P_s \square C_t) = 4t[24s - 4t - 37] - 8(s-1)(2s-3)$

(c) $LM_1(C_s \square C_t) = 16(6st - s^2 - t^2)$.

2.3 Composition

Definition 2.10. The composition $G[H]$ of graphs G and H with disjoint vertex sets and edge sets is a graph on vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever $[u_1$ is adjacent with $u_2]$ or $[u_1 = u_2$ and v_1 is adjacent with $v_2]$.

The composition is not commutative. The easiest way to visualize the composition $G[H]$ is to expand each vertex of G into a copy of H, with each edge of G replaced by the set of all possible edges between the corresponding copies of H. Hence (see [1]), the number of edges in $G[H]$ is given by $|E(G[H])| = n_1 m_2 + n_2^2 m_1$, and the degree of a vertex (u, v) of $G[H]$ is given by $d_1((u,v)/G[H]) = n_2 d_1(u/G) + d_1(v/H)$.

Lemma 2.11. Let G and H be two graphs with disjoint vertex sets with n_1 and n_2 vertices and edges sets with m_1 and m_2 edges, respectively. Then

(a) $$d_{G[H]}((u_1, v_1), (u_2, v_2)) = \begin{cases} 0, & \text{if } u_1 = u_2 \text{ and } v_1 = v_2; \\ 1, & \text{if } u_1 = u_2 \text{ and } v_1 v_2 \in E(H); \\ 2, & \text{if } u_1 = u_2 \text{ and } v_1 v_2 \notin E(H); \\ d_G(u_1, u_2), & \text{if } u_1 \neq u_2. \end{cases}$$

(b) $d_2((u,v)/G[H])) = n_2 d_2(u/G) + d_1(v/\overline{H})$.

Theorem 2.12. Let G and H be two graphs with disjoint vertex sets with n_1 and n_2 vertices and edges sets with m_1 and m_2 edges, respectively. Then

$$LM_1(G[H]) = n_2^3 LM_1(G) + n_1 M_1(H) + (2n_2^3 - 2n_2^2 - 4n_2 m_2) \sum_{u \in V(G)} d_2(u/G) + n_1(n_2-1)(n_2^2 - n_2 - 4m_2).$$

Proof: By Lemma 2.11, we obtain

$$\begin{aligned}
LM_1(G[H]) &= \sum_{(u,v)\in V(G[H])} d_2^2((u,v)/G[H]) \\
&= \sum_{(u,v)\in V(G[H])} \Big(n_2 d_2(u/G) + d_1(v/\overline{H})\Big)^2 \\
&= \sum_{u\in V(G)} \sum_{v\in V(H)} \Big(n_2^2 d_2^2(u/G) + 2n_2 d_2(u/G) d_1(v/\overline{H}) + d_1^2(v/\overline{H})\Big) \\
&= n_2^3 LM_1(G) + 2n_2[n_2(n_2-1) - 2m_2] \sum_{v\in V(G)} d_2(u/G) + n_1\Big(n_2(n_2-1)^2 \\
&\quad - 4(n_2-1)m_2 + M_1(H)\Big) \\
&= n_2^3 LM_1(G) + n_1 M_1(H) + (2n_2^3 - 2n_2^2 - 4n_2 m_2) \sum_{u\in V(G)} d_2(u/G) \\
&\quad + n_1(n_2-1)(n_2^2 - n_2 - 4m_2).
\end{aligned}$$

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As an application we present formulae for first Zagreb index of some standard graphs with bering in mind Theorem 1.1 and $LM_1(K_p) = LM_1(\overline{K_p}) = 0$ and for $p \geq 5$, $LM_1(C_p) = 4p$.

Corollary 2.13. Let G be (C_3, C_4)-free graph with n vertices and m edges. Then

- $LM_1(G[K_p]) = p^3 LM_1(G)$;
- $LM_1(K_p[G]) = p\Big(M_1(G) + n(n-1)^2 - 4m(n-1)\Big)$;
- $LM_1(G[\overline{K_p}]) = p^3 LM_1(G) + 2p^2(p-1)M_1(G) - p(p-1)(4mp - np + n)$;
- $LM_1(\overline{K_p}[G]) = pM_1(G) + pn(n-1)^2 - 4pm(n-1)$;

- $LM_1(G[C_p]) = p^3 LM_1(G) + 2p^2(p-3)(M_1(G) - 2m) + np[(p-1)(p-5)+4]$, for $p \geq 5$;
- $LM_1(C_p[G]) = pM_1(G) + 4pn^3 + pn^2(n-1)(n+3) - 4pm(5n-1)$, for $p \geq 5$.

2.4 Disjunction

Definition 2.14. The disjunction $G \vee H$ of two graphs G and H with disjoint vertex sets and edge sets is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G or v_1 is adjacent with v_2 in H.

The disjunction is commutative and the degree of a vertex (u, v) of $G \vee H$ is given by

$$d_1((u,v)/G \vee H) = n_2 d_1(u/G) + n_1 d_1(v/H) - d_1(u/G) d_1(v/H)$$

while the number of edges of $G \vee H$ is equal to $n_1^2 m_2 + n_2^2 m_1 - 2m_1 m_2$, [1].

Lemma 2.15. Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then

(a) $d_{G \vee H}((u_1, v_1), (u_2, v_2)) = \begin{cases} 0, & \text{if } u_1 = u_2 \text{ and } v_1 = v_2; \\ 1, & \text{if } u_1 u_2 \in E(G) \text{ and } v_1 v_2 \in E(H); \\ 2, & \text{otherwise.} \end{cases}$

(b) $d_2((u,v)/G \vee H) = (n_1 n_2 - 1) - n_2 d_1(u/G) - n_1 d_1(v/H) + d_1(u/G) d_1(v/H)$.

The following result which will used in proof of next our results is found in the earlier paper [10].

Theorem 2.16. Let G be a connected graph with n vertices and m edges. Then

$$LM_1(G) \leq M_1(G) + n(n-1)^2 - 4m(n-1)$$

and the equalities hold, if and only if G having diameter at most two.

In [1] the following identity was established:

Theorem 2.17. Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then

$$M_1(G \vee H) = (n_1 n_2^2 - 4n_2 m_2) M_1(G) + M_1(G) M_1(H) + (n_2 n_1^2 - 4n_1 m_1) M_1(H) + 8n_1 n_2 m_1 m_2.$$

It is clear from Lemma 2.15, if G or H not a complete graph, then $diam(G \vee H) = 2$. Hence from Theorems 2.16 and 2.17, the following result immediately follows.

Theorem 2.18. Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then
$M_1(G \vee H) = (n_1n_2^2 - 4n_2m_2)M_1(G) + M_1(G)M_1(H) + (n_2n_1^2 - 4n_1m_1)M_1(H) + 8n_1n_2m_1m_2 + n_1n_2((n_1n_2 - 1)^2 - 4(n_1n_2 - 1)(n_2^2m_1 + n_1^2m_2 - 2m_1m_2)$.

2.5 Symmetric difference

Definition 2.19. The Symmetric difference $G \oplus H$ of two graphs G and H with disjoint vertex sets and edge sets is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G or v_1 is adjacent with v_2 in H but not both.

The Symmetric difference is commutative and the degree of a vertex (u, v) of $G \otimes H$ is given by

$$d_1((u, v)/G \oplus H) = n_2d_1(u/G) + n_1d_1(v/H) - 2d_1(u/G)d_1(v/H)$$

while the number of edges of $G \oplus H$ is equal to $n_1^2m_2 + n_2^2m_1 - 4m_1m_2$, [1].

Lemma 2.20. Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then

(a) $d_{G\oplus H}((u_1, v_1), (u_2, v_2)) = \begin{cases} 0, & \text{if } u_1 = u_2 \text{ and } v_1 = v_2; \\ 1, & \text{if } u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H) \text{ but not both;} \\ 2, & \text{otherwise.} \end{cases}$

(b) $d_2((u, v)/G \oplus H) = (n_1n_2 - 1) - n_2d_1(u/G) - n_1d_1(v/H) + 2d_1(u/G)d_1(v/H)$.

In [1] the following identity was established:

Theorem 2.21. Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then

$$M_1(G \oplus H) = (n_1n_2^2 - 8n_2m_2)M_1(G) + 4M_1(G)M_1(H) + (n_2n_1^2 - 8n_1m_1)M_1(H) + 8n_1n_2m_1m_2.$$

It is clear from Lemma 2.20, if G or H not a complete graph, then $diam(G \oplus H) = 2$. Hence from Theorems 2.16 and 2.21, the following result immediately follows.

Theorem 2.22. Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then
$M_1(G \oplus H) = (n_1 n_2^2 - 8n_2 m_2)M_1(G) + 4M_1(G)M_1(H) + (n_2 n_1^2 - 8n_1 m_1)M_1(H) + 8n_1 n_2 m_1 m_2 + n_1 n_2((n_1 n_2 - 1)^2 - 4(n_1 n_2 - 1)(n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2)$.

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